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Surjective isometries of $L^1 \cap L^\infty[0,\infty)$ and $L^1 + L^\infty[0,\infty)$ *

by Ryszard Grząślewicz and Helmut H. Schaefer

Instytut Matematyki, Politechnika, PL-50-370 Wroclaw, Poland
Mathematisches Institut, Universität Tübingen, Auf der Morgenstelle 10, D-7400 Tübingen, Germany

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S. Banach in his classical book [1] characterized the surjective isometries on $L^p[0,1]$ and l^p $(1 \le p < \infty)$. Next J. Lamperti [10] extended Banach's results. Further progress in the study of isometries of various Banach function spaces was made by many authors; let us cite G. Lumer [11, 12], K.W. Tam [17], M.S. Braverman and E.M. Semenov [2]. Note that the form of isometries is also known in various concrete function spaces; for example $L_{p,1}$ ([3]), $L^1 \cap L^p$ $(1 < p < \infty)$ ([7]).

G. Gould [6] introduced the spaces $L^1 \cap L^\infty$ and $L^1 + L^\infty$. They are playing a key role in interpolation theory.

The purpose of this paper is to characterize the surjective (= invertible) isometries of $L^1 \cap L^\infty[0,\infty)$ as well as those of $L^1 + L^\infty[0,\infty)$ (we consider $[0,\infty)$ with the Lebesgue measure μ). The linear space $L^1 \cap L^\infty[0,\infty)$, equipped with the norm

$$\|f\| = \max\{\|f\|_1, \|f\|_\infty\}$$

is a Banach lattice.

The second important space in interpolation theory is the space $L^1 + L^\infty[0,\infty)$. This space consists of all functions f on $[0,\infty)$ which can be written as $g + h$ with $g \in L^1[0,\infty)$ and $h \in L^\infty[0,\infty)$, and is a Banach lattice under the norm $\|f\|_* =$

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$\inf\{\|g\|_1+\|h\|_\infty\colon\ f=g+h,\ g\in L^1[0,\infty),\ h\in L^\infty[0,\infty]\}$. Note that the space $L^1\cap L^\infty[0,\infty)$ is a proper subspace of the space dual to $L^1+L^\infty[0,\infty)$, and conversely. In the case of $L^1\cap L^\infty(X,\nu)$ where $\nu(X)<\infty$, where each atom has measure less than one and the atomless part of X has measure greater than one, the injective isometries have the Banach–Lamperti form. But in $L^1\cap L^\infty[0,\infty)$ there exists an injective isometry which does not have the Banach–Lamperti form (see [8, 9] for a counterexample).

Note that every isometry of $L^1(X,\mathbf{A},\nu)$ is of the form

$$Tf=r\Phi(f)$$

where Φ is a positive operator induced by a regular set isomorphism and the function r is such that

$$\mathbf{E}\{|r|\mid\Phi(\mathbf{A})\}=\frac{d\nu\circ\Phi^{-1}}{d\nu}$$

(see [3], [7]). Under some additional assumption on the measure space $(X,\mathbf{A},\nu)$, in view of Sikorski's theorem [16] (see also [12]), for every regular set isomorphism $\Phi:\mathbf{A}\to\mathbf{A}$ there exists $Y_0\in\mathbf{A}$ and a measurable function ϕ from Y_0 onto X such that $\Phi(A)=\phi^{-1}(A)$ for all $A\in\mathbf{A}$. Hence when T is a surjective isometry of $L^1[0,\infty)$ we have

$$(Tf)(t)=r(t)f(\phi(t))$$

where $\phi:[0,\infty)\to[0,\infty)$ is an invertible measurable transformation and $|r|=d\phi^{-1}\circ\mu/d\mu$.

In general injective isometries of L^∞-spaces do not have this nice Banach–Lamperti form. On the other hand, every surjective isometry of L^∞ is order continuous.

MAIN THEOREMS

THEOREM 1. *Let T be a surjective isometry of $L^1\cap L^\infty[0,\infty)$. Then T is of the of the form*

$$(Tf)(t)=r(t)f(\tau(t))$$

where $|r|=1$ and $\tau:[0,\infty)\to[0,\infty)$ is an invertible measure preserving transformation.

THEOREM 2. *Let T be a surjective isometry of $L^1+L^\infty[0,\infty)$. Then there exists an invertible measure preserving transformation $\tau:[0,\infty)\to[0,\infty)$ such that*

$$(Tf)(t)=r(t)f(\tau(t))$$

where $|r|=1$.

COROLLARY. *Every surjective isometry of $L^1+L^\infty[0,\infty)$ necessarily induces an isometry on both $L^1[0,\infty)$ and $L^\infty[0,\infty)$, as well as on the order continuous dual $L^1\cap L^\infty[0,\infty)$, with respect to the appropriate norms, and conversely.*

The proof of Theorem 1 is based on the following two lemmas.

LEMMA 1. *Let T be an isometry of $L^1 \cap L^\infty[0,\infty)$ (not necessarily surjective). Suppose that there exists $h \in L^1 \cap L^\infty$ such that $\|Th\|_\infty < \|Th\| = \|h\| = \|h\|_1$. Then T is also an L^1-isometry.*

PROOF. We may assume that $\mu((\operatorname{supp} h)^c) \geq 1$. If not we can find A such that $\mu(A) \geq 1$ and $\|1_A h\| < (\|Th\| - \|Th\|_\infty)/4$, and we take $g \in L^1 \cap L^\infty[0,1)$ such that $\operatorname{supp} g$ is included in A^c and $\|g\| < (\|Th\| - \|Th\|_\infty)/4$, $\|1_{A^c}h + g\|_1 = \|h\|_1$, and $\|1_{A^c}h + g\|_\infty \leq \|h\|_\infty$. And now instead of h we consider $1_{A^c}h + g$ (indeed, we have $\|T(1_{A^c}h+g)\|_\infty \leq \|Th\|_\infty + \|T1_A h\| + \|Tg\| < (\|Th\| + \|Th\|_\infty)/2 < \|Th\|_1 - \|T1_A h\| - \|Tg\| \leq \|T(1_{A^c}h+g)\|_1$).

Choose a positive $\varepsilon < \min\{1, \|Th\| - \|Th\|_\infty\}$. Let A, B be disjoint subsets of $(\operatorname{supp} h)^c$ such that $\mu(A \cup B) < \varepsilon$. We have $\|Th \pm \varepsilon T1_A \pm \varepsilon T1_B\|_\infty \leq \|Th\|_\infty + \varepsilon\,\|T(1_A \pm 1_B)\|_\infty \leq \|Th\|_\infty + \varepsilon < \|Th\| < \|h\| + \varepsilon\mu(A \cup B) = \|h \pm \varepsilon 1_A \pm 1_B\|_1 \leq \|h \pm \varepsilon 1_A \pm \varepsilon 1_B\| = \|Th \pm \varepsilon(T1_A \pm T1_B)\|$. Hence $\|Th \pm \varepsilon(T1_A \pm T1_B\|_1 = \|Th \pm \varepsilon(T1_A \pm T1_B)\| = \|h \pm \varepsilon 1_A \pm \varepsilon 1_B\| = \|h \pm \varepsilon 1_A \pm \varepsilon 1_B\|_1 = \|h\|_1 + \varepsilon(\|1_A\|_1 + \|1_B\|_1)$. Therefore $\|T1_A\|_1 = \|1_A\|_1$, $\|T1_B\|_1 = \|1_B\|_1$, and $\operatorname{supp} T1_A \cap \operatorname{supp} T1_B = \emptyset$ so $\|T1_A\|_1 = \|1_A\|_1$ for all $A \subset (\operatorname{supp} h)^c$. For subsets of $\operatorname{supp} h$ we consider instead of h the function 1_A, where $A \subset (\operatorname{supp} h)$ and $\mu(A) > 1$ (then $1 = \|1_A\|_\infty < \|1_A\|$). We have $\|T1_A\|_1 > 1 = \operatorname{supp}_{1 \leq i \leq n} \|T1_{A_i}\|_\infty = \|T1_A\|_\infty$ where $\{A_i\}$ is a partition of A with $\mu(A_i) < \varepsilon$ (then $\operatorname{supp} T1_{A_i}$ are disjoint). Therefore T is an L^1-isometry.

LEMMA 2. *Let T be a surjective isometry of $L^1 \cap L^\infty[0,\infty)$. Then T is also an L^1-isometry.*

PROOF. Let $A \subset [0,\infty)$ be such that $\mu(A) = 2$. We have $\|T1_A\| = \|1_A\| = \|1_A\|_1 = 2$. Suppose, to get a contradiction, that $\|T1_A\|_1 < \|1_A\|_1 = 2$. Then we have $\|T1_A\|_\infty = 2$. We can find $f_1, f_2 \in L^1 \cap L^\infty[0,\infty)$ such that $f_1 + f_2 = T1_A$ and

$$\|f_i\|_1 < \|f_i\|_\infty = 1 \quad i = 1,2.$$

Put $g_i = T^{-1} f_i$. Obviously $g_1 + g_2 = 1_A$ and $\|g_1 + g_2\|_1 = 2$. Since $\|g_i\|_1 \leq \|g_i\| = 1$, we get $\|g_i\|_1 = 1$ and $\operatorname{supp} g_i \subset A$.

Moreover $\|g_i\|_1 \leq \|g_i\| = 1$. We can find u such that $\|u\| < 1 - \|f_i\|_1$ and $\|g_2 + u\|_\infty < 1$, $\|g_1 - u\|_\infty < 1$, $\|g_1 + u\|_1 = \|g_2 - u\|_1 = 1$. Put $h_1 = f_1 + Tu$, $h_2 = f_2 - Tu$.

We have

$$\|h_1\|_1 < 1 = \|g_i - (-1)^i u\| = \|h_i\| = \|h_i\|_\infty.$$

Hence the isometry T^{-1} has the property required in Lemma 1, i.e. there exists h_1 such that

$$\|T^{-1}h_1\|_\infty = \|g_1 + u\|_\infty < 1 = \|T^{-1}h_1\| = \|h_1\|_1.$$

Hence by Lemma 1 T^{-1} is an L^1-isometry, and also is T.

PROOF OF THEOREM 1. Let T be a surjective isometry of $L^1 \cap L^\infty[0,\infty)$. Then by Lemma 2 T is an L^1-isometry on $L^1 \cap L^\infty[0,\infty)$ and we can extend T to an isometry on $L^1[0,\infty)$. Using the Banach-Lamperti characterization of isometries and the Sikorski's Theorem we can find an invertible measurable transformation $\tau: [0,\infty) \to [0,\infty)$ such that

$$(Tf)(t) = r(t) f(\tau(t))$$

where $|r| = d\tau^{-1} \circ \mu / d\mu$. Suppose, to get a contradiction, that $|r| > 1$ for all $t \in A$, where $\mu(A) > 0$. We may and do assume that $\mu(\tau^{-1}(A)) < 1$. Then $\|1_A\| = \|1_A\|_\infty = 1 < \|T1_{\tau^{-1}(A)}\|_\infty \le \|T1_{\tau^{-1}(A)}\|$, so $|r| \le 1$. Since T is invertible, $r(t) \ne 0$ for all t. Hence in the case when $|r(t)| < 1$ we can use the above argument for the isometry $T^{-1}f(t) = (1/r(t)) f(\tau^{-1}(t)))$. Hence $|r| = 1$, i.e. τ is measure preserving.

PROOF OF THEOREM 2. We divide the proof into several steps.

We denote by $B = [f \in L^1 + L^\infty[0,\infty): \|f\|_* \le 1\}$ the unit ball of $L^1 + L^\infty[0,\infty)$. Put

$$U = \{f \in L^1: \|f\|_1 \le 1\}, \qquad V = \{g \in L^\infty[0,\infty): \|g\|_\infty \le 1\}.$$

Step 1. The set U is $\sigma(L^1 + L^\infty[0,\infty), L^1 \cap L^\infty[0,\infty))$-closed and the set V is $\sigma(L^1 + L^\infty[0,\infty), L^1 \cap L^\infty[0,\infty))$-compact.

Indeed, suppose that f is an element of the $\sigma(L^1 + L^\infty[0,\infty), L^1 \cap L^\infty[0,\infty))$-closure of U. Then there exists a filter Φ on U such that $\Phi \to f$ for the $\sigma(L^1 + L^\infty[0,\infty), L^1 \cap L^\infty[0,\infty))$-topology. For given $\varepsilon > 0$ and $g \in L^1 \cap L^\infty[0,\infty)$ there exists $F \in \Phi$ such that $\int (f-h) g \, d\mu \le \varepsilon$ for all $h \in F$. Take $g = 1_{[0,a]} \overline{\operatorname{sgn} f}$. Obviously $g \in L^1 \cap L^\infty[0,\infty)$. We have

$$\int_0^a |f| \, d\mu = \int fg \, d\mu \le \varepsilon + \int hg \, d\mu \le \varepsilon + \int |h| \, d\mu < 1 + \varepsilon.$$

Thus $\|f\|_1 \le 1 + \varepsilon$, so $\|f\|_1 \le 1$, i.e. $f \in U$. This shows that U is $\sigma(L^1 + L^\infty[0,\infty), L^1 \cap L^\infty[0,\infty))$-closed.

Note that V is compact because the $\sigma(L^1 + L^\infty[0,\infty), L^1 \cap L^\infty[0,\infty))$-topology is a Hausdorff topology weaker than the w^*-topology (namely $\sigma(L^\infty[0,\infty), L^1[0,\infty))$ on V, the unit ball of $L^\infty[0,\infty)$.

Step 2. The set $\operatorname{conv}(U \cup V)$ is $\sigma(L^1 + L^\infty[0,\infty), L^1 \cap L^\infty[0,\infty))$-closed, hence norm-closed in $L^1 + L^\infty[0,\infty)$. Thus $\operatorname{conv}(U \cup V)$ coincides with the closed unit ball B of $L^1 + L^\infty[0,\infty)$.

Indeed, it is easy to see that $B = \overline{\operatorname{conv}}(U \cup V)$. Thus it is enough to show that $\operatorname{conv}(U \cup V)$ is $\sigma(L^1 + L^\infty[0,\infty), L^1 \cap L^\infty[0,\infty))$-closed.

Let (z_α) be a net of elements of the convex ball $\operatorname{conv}(U \cup V)$ which converges to some z in the $\sigma(L^1 + L^\infty[0,\infty), L^1 \cap L^\infty[0,\infty))$-topology. For every α there exist $\lambda_\alpha \in [0,1)$, $v_\alpha \in V$, $u_\alpha \in U$ such that $z_\alpha = \lambda_\alpha u_\alpha + (1-\lambda_\alpha) v_\alpha$. Because $[0,1]$ and V are compact we can find subnets (z_β), (λ_b), (u_β), (v_β) such that $\lambda_\beta \to \lambda$, $v_\beta \to v \in V$ and obviously $z_\beta \to z$. If $\lambda = 0$ then $\lambda_\beta u_\beta \to 0$ since U is bounded, hence $z = v$. If $\lambda > 0$ then $\lambda_\beta u_\beta = z_\beta - (1-\lambda_\beta) v_\beta \to z - (1-\lambda) v =: \lambda u$.

Hence $u \in U$ because U by Lemma 3 is closed. So we obtain that $z = \lambda u + (1+\lambda)v$, $u \in U$, $v \in V$, i.e. $\operatorname{conv}(U \cup V)$ is $\sigma(L^1 + L^\infty[0,\infty), L^1 \cap L^\infty[0,\infty))$-closed.

Step 3. $\operatorname{ext} V = \operatorname{ext} B = \{f \in L^1 + L^\infty[0,\infty) \colon |f| = 1\}$.

Indeed, obviously $f \in V$ if and only if $|f(t)| \leq 1$ for almost all t. Hence the conditions $|f(t)| = 1$ and $f \pm g \in V$ imply $g(t) = 0$. In the case when $\mu(\{t \colon |f(t)| < 1\}) > 0$ there exists $\varepsilon > 0$ and a measurable set A with positive measure such that $|f \pm \varepsilon 1_A| \leq 1$. This shows that $\operatorname{ext} V = \{f \in L^1 + L^\infty[0,\infty)\colon |f| = 1\}$.

Let $f \in \operatorname{ext} B$ and put $C_\varepsilon = \{t \colon |f(t)| > 1 + \varepsilon\}$. Suppose that $\mu(C_0) > 0$. Then there exists $\varepsilon > 0$ such that $\mu(C_\varepsilon) > 0$. We divide the set C_ε into two disjoint subsets E_1. E_2 of equal measure. Then it is not difficult to check that $\|f \pm \varepsilon(1_{E_1} - 1_{E_2}) \operatorname{sgn} f\|_* \leq 1$. This holds because the values of the functions $f \pm \varepsilon(1_{E_1} - 1_{E_2}) \operatorname{sgn} f$ for t belonging to $C_\varepsilon = E_1 \cup E_2$ are in fact essential for the calculation of their norms.

Put $D_\varepsilon = \{t \colon |f(t)| < 1 - \varepsilon\}$. Now suppose that $|f| \leq 1$ and $\mu(D_0) > 0$. Then $f^*(1) = 1$. Fix $\varepsilon > 0$ such that $\mu(D_\varepsilon) > 0$. Then $|f \pm \varepsilon 1_{D_\varepsilon}| \leq 1$, so $\|f \pm \varepsilon 1_{D_\varepsilon}\|_* \leq 1$. This shows that if $f \in \operatorname{ext} B$ then $\mu(C_0) = \mu(D_0) = 0$, i.e. that $|f| = 1$. This completes the proof of Step 3.

Next we note that for each $f \geq 0$, the order interval $[0,f]$ is the convex norm closure of $\operatorname{ext}[0,f]$, the set of all its extreme points (see [14, II, Exercise 4(f) p. 143]). Because in the case of real scalars V is the order interval $([-\mathbf{1},\mathbf{1}] = 2[\mathbf{0},\mathbf{1}] - \mathbf{1})$, the previous result applies. In the complex case, we approximate the function $|f|$ and then multiply by $\operatorname{sgn} f$. Thus we obtain:

Step 4. $V = \overline{\operatorname{conv}} \operatorname{ext} V$ in $L^1 + L^\infty[0,\infty)$ (in the norm topology).

For a Banach lattice E we denote by E_n^* the vector space (band) of all order continuous linear functionals on E (see [14, II.4 or [18], 85)] for definitions and basic facts).

Step 5. Let E be a Banach lattice, separated by E_n^*, and let $\mathscr{T}$ be a (norm) dense ideal in E. If $T \colon E \to E$ is order bounded on E (in particular, if $T \geq 0$) and order continuous on $\mathscr{T}$, then T is order continuous on E.

Indeed, suppose that $x_a \downarrow 0$ in E, say $x_a \leq x_0$). Fix $\varepsilon > 0$. Then there exists $y \in \mathscr{T}$ such that $0 \leq y \leq x_0$ and $\|x_0 - y\| < \varepsilon$. Put $\bar{x}_a := x_a \wedge y \in \mathscr{T}$. Then $0 \leq x_a - \bar{x}_a = x_a \wedge x_0 - x_a \wedge y \leq x_0 - y$, so $\|x_a - \bar{x}_a\| \leq \|x_0 - y\| < \varepsilon$ for all a. We have $|Tx_a| \leq |T\bar{x}_a| + |T(x_a - \bar{x}_a)|$. Now, for $0 \leq f \in E_n^*$ we get $f(|Tx_a|) \leq f(|T\bar{x}_a|) + \|T\|\|f\|\|x_a - \bar{x}_a\| \leq f(|T\bar{x}_a|) + \|T\|\|f\|\varepsilon$. This implies that $\inf_a f(|Tx_a|) \leq \|T\|\|f\|\varepsilon$, so $\inf_a f(|Tx_a|) = 0$.

From [15, theorem A] and the assumption that E is separated by E_n^* it follows that the positive cone $E_+ = \{x \in E \colon x \geq 0\}$ is $\sigma(E, \mathscr{T})$-closed. Hence we obtain $\inf |Tx_a| = 0$, which ends the proof.

Step 6. Let $T \colon L^1 + L^\infty[0,\infty) \to L^1 + L^\infty[0,\infty)$ be a surjective isometry. Then $T^*(L^1 \cap L^\infty[0,\infty)) = L^1 \cap L^\infty[0,\infty)$.

Indeed, let $T \colon L^1 + L^\infty[0,\infty) \to L^1 + L^\infty[0,\infty)$ be a surjective isometry. From Step 3 and Step 4 it follows that $T_0 := T|_{L^\infty[0,\infty)}$ is a surjective isometry of

$L^\infty[0,\infty)$. If $L^\infty[0,\infty)$ is represented as $C(X)$ (X compact), then $T_0 f = hf\circ\phi$ (ϕ homeomorphism of X, $|h|=1$). In particular, T_0 is order continuous; now consider on $L^1+L^\infty[0,\infty)$ the operator $\bar{T}:=\tilde{h}^{-1}T$, where $\tilde{h}\in L^\infty$ corresponds to $h\in C(X)$. Since $L^\infty[0,\infty)$ is a dense ideal in $L^1+L^\infty[0,\infty)$, it follows that $\bar{T}$ is a positive isometry of $L^1+L^\infty[0,\infty)$. It is well known that for $E=L^1+L^\infty[0,\infty)$, the space E_n^* equals $L^1\cap L^\infty[0,\infty)$. Because $\bar{T}$ is order continuous by Step 5, so is T. We get that T^* maps E_n^* into E_n^*. Now we apply this to $(T^{-1})^*=(T^*)^{-1}$ and we obtain $T^*(L^1\cap L^\infty[0,\infty))=L^1\cap L^\infty[0,\infty)$ isometrically. In view of Thm. 1, this ends the proof of Thm. 2.

REMARK. Step 5 implies that surjective isometries of $L^1+L^\infty[0,\infty)$ are continuous for the $\sigma(L^1+L^\infty[0,\infty),\ L^1\cap L^\infty[0,\infty))$-topology.

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